

DEFORMATIONS OF CALABI-YAU HYPERSURFACES ARISING FROM DEFORMATIONS OF TORIC VARIETIES

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0. INTRODUCTION.

It is a well known fact now that the infinitesimal deformations of Calabi-Yau manifolds are unobstructed (see [Ti, To]). A generalization of this theorem for Calabi-Yau orbifolds nonsingular in codimension two is due to Z. Ran (see [R]). There are easy “polynomial” deformations of Calabi-Yau hypersurfaces in toric varieties performed by changing the coefficients of the defining polynomial of the hypersurface. But according to the unobstructedness and the calculation of the space of infinitesimal deformations (e.g., [M2]), there must also exist “non-polynomial” deformations of the minimal Calabi-Yau hypersurfaces leading outside the ambient toric variety. In this paper, we have constructed the missing non-polynomial deformations, which are explicitly described as abstract Calabi-Yau varieties.

The first examples of non-polynomial deformations mentioned in [CadFKMo] were constructed by Sheldon Katz and David Morrison for two parameter models (e.g., for crepant resolutions of degree 8 hypersurfaces in $\mathbb{P}(1, 1, 2, 2, 2)$). Despite the importance of these deformations in string theory (see [KMoP], [KaKLMc]), a general solution was elusive for quite a while. All of the previous constructions of non-polynomial deformations were realized by embedding an ample Calabi-Yau hypersurface of some weighted projective space into another variety where a deformation was performed, and an appropriate blow-up of this deformation produced the deformation of the crepant resolution. Our approach is different. First, we found a credible evidence that the non-polynomial deformations of the Calabi-Yau hypersurfaces should be induced by the deformations of the ambient toric varieties. Second, the nature of the definition of a toric variety led us to the idea to reglue the affine toric varieties in order to deform the complex structure on the ambient variety. A further investigation revealed us that this deformation is obtained by an automorphism of an open toric subvariety corresponding to its root. Finally, the

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deformation gives a flat family of hypersurfaces changing the complex structure on the original Calabi-Yau hypersurface.

We have the following plan for the paper. Section 1 gives a set up of the toric varieties in terms of homogeneous coordinates, and also describes the space of infinitesimal deformations of minimal Calabi-Yau hypersurfaces. Then, Section 2 has the construction of deformations of complete toric varieties with a big and nef divisor. We show that such deformations induce deformations of the hypersurfaces as well. In Section 3, we calculate the infinitesimal deformations in terms of Čech cocycles corresponding to the global deformations of the hypersurfaces. For a basis of the space of infinitesimal deformations of the minimal Calabi-Yau hypersurface, we get the corresponding global deformations.

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1. INFINITESIMAL DEFORMATIONS OF CALABI-YAU HYPERSURFACES.

This section recalls the definition of a toric variety as a quotient of a Zariski open subset of an affine space by a linear diagonal action of some algebraic subgroup of a complex torus. We also review the description of the space of infinitesimal deformations $H^1(X, \mathcal{T}_X)$ for a minimal Calabi-Yau hypersurface X in a complete simplicial toric variety \mathbf{P}_Σ . Then, we show that the non-polynomial infinitesimal deformations represented by Čech cocycles have a “lift” to the space $H^1(\mathbf{P}_\Sigma, \mathcal{T}_{\mathbf{P}_\Sigma})$. This leads us to the crucial observation that the non-polynomial deformations of a Calabi-Yau hypersurface should be induced by the deformations of the ambient toric variety. Basic facts about toric varieties can be found in [C2, F, D, Od].

We start defining toric varieties with a lattice N of rank d . A rational convex polyhedral cone $\sigma \subset N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ is a cone generated by finitely many elements $u_1, \dots, u_s \in N$:

$$\sigma = \{\lambda_1 u_1 + \dots + \lambda_s u_s : \lambda_1 \dots \lambda_s \geq 0\}.$$

Such a cone σ is strongly convex if $\sigma \cap (-\sigma) = 0$. A face of σ is the intersection of σ with one of its supporting hyperplanes: $\{L = 0\} \cap \sigma$, where L is a linear form, nonnegative on σ . A *fan* Σ is defined to be a finite collection of strongly convex rational polyhedral cones in $N_{\mathbb{R}}$ such that

- (i) Each face of a cone in Σ belongs to Σ ,
- (ii) The intersection of two cones in Σ is a face of each.

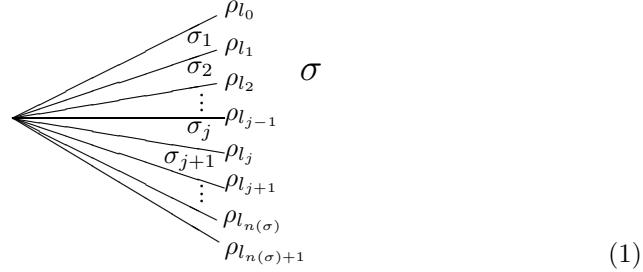
Denote by $\Sigma(k)$ the set of k -dimensional cones of a fan Σ . Suppose that 1-dimensional cones of Σ span $N_{\mathbb{R}}$. Then, by [C1], the toric variety \mathbf{P}_Σ , associated with the fan Σ , can be constructed as follows. Let $\mathbb{A}^n = \text{Spec}(\mathbb{C}[x_1, \dots, x_n])$, where n is the number of 1-dimensional cones $\Sigma(1)$, and let $B(\Sigma) = \langle \prod_{\rho_i \notin \sigma} x_i \rangle$ be the ideal in $\mathbb{C}[x_1, \dots, x_n]$. Then, the complement to the closed subset defined by $B(\Sigma)$ gives the Zariski open set $\mathbb{A}^n \setminus \mathbf{V}(B(\Sigma))$. This set is invariant under the action of an affine algebraic D -group G , a subgroup of $(\mathbb{C}^*)^n$, defined as a kernel of the group homomorphism $(\mathbb{C}^*)^n \rightarrow (\mathbb{C}^*)^d$ sending (x_1, \dots, x_n) to $(\prod_{i=1}^n x_i^{\langle m_1, e_i \rangle}, \dots, \prod_{i=1}^n x_i^{\langle m_d, e_i \rangle})$, where e_1, \dots, e_n are the minimal lattice generators of the 1-dimensional cones ρ_1, \dots, ρ_n of the fan Σ , and where m_1, \dots, m_d is a basis of the dual lattice $M = \text{Hom}(N, \mathbb{Z})$ and $\langle \cdot, \cdot \rangle$ is the pairing. Theorem 2.1 in [C1], allows us to define the toric variety \mathbf{P}_Σ , associated with the fan Σ , as the categorical quotient of $\mathbb{A}^n \setminus \mathbf{V}(B(\Sigma))$ by G . The ring $S = \mathbb{C}[x_1, \dots, x_n]$ is called

the homogeneous coordinate ring of \mathbf{P}_Σ , where the variables x_1, \dots, x_n give the irreducible torus $(\mathbb{C}^*)^n/G$ -invariant divisors D_1, \dots, D_n . This ring is graded by the Chow group $A_{d-1}(\mathbf{P}_\Sigma)$, assigning $[\sum_{i=1}^n a_i D_i]$ to $\deg(\prod_{i=1}^n x_i^{a_i})$.

To describe the infinitesimal deformations we first consider a more general situation than the one of a Calabi-Yau hypersurface. As in [M2, Section 4], let X be a big and nef hypersurface, defined by $f \in S_\beta$, in the toric variety \mathbf{P}_Σ . Then, by Proposition 1.2 in [M1], there is the associated toric morphism $\pi : \mathbf{P}_\Sigma \rightarrow \mathbf{P}_{\Sigma_X}$. Consider a 2-dimensional cone $\sigma \in \Sigma_X$ with at least one 1-dimensional cone $\rho \subset \sigma$, whose generator lies in the relative interior $\text{int}(\sigma)$ of σ . Using such a cone σ form an open covering of the toric variety \mathbf{P}_Σ by the sets

$$U_{\sigma'} = \left\{ x \in \mathbf{P}_\Sigma : \prod_{\rho_i \subset \sigma \setminus \sigma'} x_i \neq 0 \right\}$$

for all cones $\sigma' \in \Sigma(2)$ that lie in σ . Fix an order for this open covering corresponding to as the cones lie inside σ :



where $n(\sigma)$ is the number of cones ρ such that $\rho \subset \sigma$ and $\rho \notin \Sigma_X(1)$.

Now let \mathbf{P}_Σ be a simplicial toric variety and assume in addition that the hypersurface $X \subset \mathbf{P}_\Sigma$ is quasismooth, i.e., the partial derivatives $\partial f / \partial x_1, \dots, \partial f / \partial x_n$ do not vanish simultaneously on \mathbf{P}_Σ . Consider a refinement $U_{i,\sigma_j} = U_i \cap U_{\sigma_j}$ of the above open covering and the open covering $\mathcal{U} = \{U_i\}_{i=1}^n$, where $U_i = \{x \in \mathbf{P} : f_i(x) \neq 0\}$ and $f_i := \partial f / \partial x_i$. Denote the refined covering \mathcal{U}^σ , considering the order on this covering as the lexicographic order for the pairs of indices (i, j) .

In [M2], we had two different types of Čech cocycles which represent some elements in $H^1(X, \mathcal{T}_X)$:

Definition 1.1. Given $A \in S_\beta$, set

$$(\gamma_A)_{i_0 i_1} = \left\{ \frac{\sqrt{-1} A \langle \partial_{i_0} \wedge \partial_{i_1}, df \rangle}{f_{i_0} f_{i_1}} \right\}_{i_0 i_1},$$

where $\partial_i := \frac{\partial}{\partial x_i}$ and $\langle \cdot, \cdot \rangle$ denotes the contraction.

Given $\rho_l \subset \sigma \in \Sigma_X(2)$ such that $\rho_l \notin \Sigma_X(1)$, then, as in (1), $l = l_j$ for some j , and we set

$$\partial_j^l = \frac{x_{l_{j-1}} \partial_{l_{j-1}}}{\text{mult}(\sigma_j)}, \quad \partial_{j+1}^l = -\frac{x_{l_{j+1}} \partial_{l_{j+1}}}{\text{mult}(\sigma_{j+1})}, \text{ and } \partial_k^l = 0 \text{ for } k \neq j, j+1.$$

For $B \in S_{\beta_1^\sigma}$ (here, $\beta_1^\sigma := \sum_{\rho_i \subset \sigma} \deg(x_i)$), define

$$(\gamma_B^l)_{(i_0, j_0)(i_1, j_1)} = \left\{ \frac{B}{\prod_{\rho_i \subset \sigma} x_i} \left(\frac{\langle \partial_{i_1} \wedge \partial_{j_1}^l, df \rangle}{f_{i_1}} - \frac{\langle \partial_{i_0} \wedge \partial_{j_0}^l, df \rangle}{f_{i_0}} \right) \right\}_{(i_0, j_0), (i_1, j_1)}.$$

The above cocycles give maps: $\gamma_- : S_\beta \rightarrow H^1(X, \mathcal{T}_X)$ and $\gamma_-^l : S_{\beta_1^\sigma} \rightarrow H^1(X, \mathcal{T}_X)$. The following statement is a part of Theorem 7.1 in [M2].

Proposition 1.2. *Let $X \subset \mathbf{P}_\Sigma$ be a semiample anticanonical nondegenerate (i.e., transversal to the torus orbits) hypersurface in a complete simplicial toric variety, defined by $f \in S_\beta$. Then there is an isomorphism*

$$\gamma_- \oplus (\oplus_l \gamma_-^l) : R_1(f)_\beta \oplus \left(\bigoplus_{\substack{\sigma \in \Sigma_X(2) \\ e_l \in \text{int}(\sigma)}} (S/\langle x_l \rangle)_{\beta_1^\sigma} \right) \cong H^1(X, \mathcal{T}_X),$$

where the sum $\oplus_l \gamma_-^l$ is over $\rho_l \subset \sigma \in \Sigma_X(2)$ such that $\rho_l \notin \Sigma_X$, and $R_1(f) = S/J_1(f)$ with $J_1(f) := \langle x_1(\partial f / \partial x_1), \dots, x_n(\partial f / \partial x_n) \rangle : x_1 \cdots x_n$.

Remark 1.3. In the above proposition, if the Calabi-Yau hypersurface X in \mathbf{P}_Σ has terminal singularities, as in the case of a maximal projective subdivision of the fan of a Fano toric variety (see [B]), then X is a minimal Calabi-Yau orbifold (see Definition 1.4.1 in [CK]), and, by Proposition A.4.2 in [CK], the space $H^1(X, \mathcal{T}_X)$ classifies infinitesimal deformations.

The space $R_1(f)_\beta$ in Proposition 1.2 represents the polynomial deformations of a semiample nondegenerate Calabi-Yau hypersurface performed inside the toric variety. Indeed, the space $H^0(X, \mathcal{N}_{X/\mathbf{P}_\Sigma})$ classifies the infinitesimal deformations of X as a subvariety of \mathbf{P}_Σ (see [H, Chapter III, Exercise 9.7]). On the other hand, the short exact sequence

$$0 \rightarrow \mathcal{T}_X \rightarrow i^* \mathcal{T}_{\mathbf{P}_\Sigma} \rightarrow \mathcal{N}_{X/\mathbf{P}_\Sigma} \rightarrow 0$$

(here, $i : X \subset \mathbf{P}_\Sigma$ is the inclusion) gives the maps

$$H^0(X, \mathcal{N}_{X/\mathbf{P}_\Sigma}) \rightarrow H^1(X, \mathcal{T}_X) \rightarrow H^1(X, i^* \mathcal{T}_{\mathbf{P}_\Sigma}).$$

As remarked after Definition 3.1 in [M2, Section 3], the restriction of the cocycle $(\gamma_A)_{i_0 i_1}$ to $C^p(\mathcal{U}|_X, i^* \wedge^p \mathcal{T}_\mathbf{P})$ is a Čech coboundary. By the exactness of the above sequence at the middle term, we deduce that γ_A represent the polynomial deformations, i.e., those that can be performed inside the ambient space. We will now show that the other part — the non-polynomial infinitesimal deformations — in $H^1(X, \mathcal{T}_X)$ have a “lift” to $H^1(\mathbf{P}_\Sigma, \mathcal{T}_{\mathbf{P}_\Sigma})$. The images of γ_B^l , for $B \in S_{\beta_1^\sigma}$, in $H^1(X, i^* \mathcal{T}_{\mathbf{P}_\Sigma})$ are represented by

$$\left\{ \frac{B}{\prod_{\rho_i \subset \sigma} x_i} \left(\partial_{j_1}^l - \partial_{j_0}^l - \frac{\langle \partial_{j_1}^l, df \rangle}{f_{i_1}} \partial_{i_1} + \frac{\langle \partial_{j_0}^l, df \rangle}{f_{i_0}} \partial_{i_0} \right) \right\}_{(i_0, j_0)(i_1, j_1)}.$$

Since the polynomials $\langle \partial_{j_0}^l, df \rangle$ are divisible by all x_i such that e_i is in the relative interior of σ , this cocycle is equivalent up to a coboundary to

$$\left\{ \frac{B}{\prod_{\rho_i \subset \sigma} x_i} \left(\partial_{j_1}^l - \partial_{j_0}^l \right) \right\}_{j_0 j_1}, \quad (2)$$

which has an obvious lift by the restriction map $H^1(\mathbf{P}_\Sigma, \mathcal{T}_{\mathbf{P}_\Sigma}) \rightarrow H^1(X, i^* \mathcal{T}_{\mathbf{P}_\Sigma})$. Notice that the lift is independent of the hypersurface X , which is a strong evidence that the non-polynomial deformations are induced by the deformation of the ambient variety. We should also point out that the above cocycles are coboundaries in the case $B \in S_{\beta_1^\sigma}$ is divisible by x_l .

2. DEFORMATIONS OF TORIC VARIETIES AND SEMIAMPLE HYPERSURFACES.

In this section, we take the open covering of the toric variety used for the cocycles representing the non-polynomial deformations and reglue the open sets in a certain way so that the complex structure deforms on the toric variety. The construction of a flat family corresponding to this gluing is automatic. Then, we also find a subfamily which gives a non-polynomial deformation of semiample hypersurfaces. We consider complete toric varieties over complex numbers, but everything holds over an algebraically closed field.

Let \mathbf{P}_Σ be a complete toric variety with a big and nef divisor class $[X] \in A_{d-1}(\mathbf{P}_\Sigma)$. Take the open covering $\{U_{\sigma_j}\}_{j=1}^{n(\sigma)}$, considered in Section 1 for $\sigma \in \Sigma_X(2)$ with at least one e_l , $l = l_j$ as in (1), lying in its relative interior. It was practical to use this open covering in [M2] for calculations, but we should note that this covering of the toric variety \mathbf{P}_Σ is a refinement of a covering by two open sets U_0^l and U_1^l given by $\prod_{k>j} x_{l_k} \neq 0$ and $\prod_{k<j} x_{l_k} \neq 0$, respectively. These sets intersect in the open toric subvariety

$$U_0^l \cap U_1^l = \{x \in \mathbf{P}_\Sigma : \prod_{\rho_l \neq \rho_i \subset \sigma} x_i \neq 0\}.$$

Next, note that the noncomplete toric variety $U_0^l \cap U_1^l$ has the following automorphisms. The monomials $\prod_{\rho_i \subset \sigma} x_i \prod_{i=1}^n x_i^{\langle u, e_i \rangle}$ in $S_{\beta_1^\sigma}$, where $\beta_1^\sigma = \sum_{\rho_i \subset \sigma} \deg(x_i)$, correspond to $u \in M$ such that $\langle u, e_i \rangle \geq -1$ for $\rho_i \subset \sigma$ and $\langle u, e_i \rangle \geq 0$ for $\rho_i \not\subset \sigma$. Then the monomials $x_l \prod_{i=1}^n x_i^{\langle u, e_i \rangle}$ have the same degree as x_l and do not have poles in the intersection $U_0^l \cap U_1^l$. The 1-parameter subgroups of automorphisms $\text{Aut}(U_0^l \cap U_1^l)$ are induced by

$$y_\lambda(x_1, \dots, x_l, \dots, x_n) = (x_1, \dots, x_l + \lambda x_l \prod_{i=1}^n x_i^{\langle u, e_i \rangle}, \dots, x_n). \quad (3)$$

For those u above, which satisfy $\langle u, e_l \rangle = -1$, the lattice point $-u$ is called a *root* of the noncomplete toric variety $U_0^l \cap U_1^l$ (this is similar to the definitions in [C1] and [De]).

To deform the complex structure on the toric variety \mathbf{P}_Σ , we reglue the open subsets U_0^l and U_1^l of \mathbf{P}_Σ along $U_0^l \cap U_1^l$ by the open immersions:

$$U_0^l \cap U_1^l \hookrightarrow U_k^l, \quad k = 1, 2,$$

which send a point in $U_0^l \cap U_1^l$ represented by $(x_1, \dots, x_l, \dots, x_n)$ to the corresponding points in U_k^l represented by

$$(x_1, \dots, x_l + (-1)^k \lambda x_l \prod_{i=1}^n x_i^{\langle u, e_i \rangle}, \dots, x_n).$$

All this is well defined because there are natural commutative diagrams:

$$\begin{array}{ccc}
 \{x \in \mathbb{A}^n \setminus \mathbf{V}(B(\Sigma)) : \prod_{k < j} x_{l_k} \neq 0\} & \longrightarrow & U_0^l \\
 \uparrow & & \uparrow \\
 \{x \in \mathbb{A}^n \setminus \mathbf{V}(B(\Sigma)) : \prod_{\rho_l \neq \rho_i \subset \sigma} x_i \neq 0\} & \longrightarrow & U_0^l \cap U_1^l \\
 \downarrow & & \downarrow \\
 \{x \in \mathbb{A}^n \setminus \mathbf{V}(B(\Sigma)) : \prod_{k > j} x_{l_k} \neq 0\} & \longrightarrow & U_1^l,
 \end{array}$$

where the horizontal arrows are toric morphisms (see [C1, page 27]) and the vertical arrows are the open immersions. If $-u$ is *not* a root of $U_0^l \cap U_1^l$, it is not hard to see that $\prod_{i=1}^n x_i^{\langle u, e_i \rangle}$ will have no poles on one of the open sets U_0^l or U_1^l , whence (3) induces an automorphism on that set, and the above gluing produces the same variety \mathbf{P}_Σ . If $-u$ is the root, the glued set is not a toric variety any more, but it still has a Zariski open covering by noncomplete toric varieties, in fact, the same covering as the initial complete toric variety had. Such varieties fit into the category of toroidal varieties (see [D]).

The description of the deformation of complex structure we presented is analogous to Spencer's idea on the complex deformation in [Kos]. However, algebraic geometry defines a (*global*) *deformation* of a scheme as a flat family with one of the fibers isomorphic to the scheme. The construction of such a flat family from the gluing condition is straightforward. In our situation, consider two sets $U_0^l \times \mathbb{A}^1$ and $U_1^l \times \mathbb{A}^1$, which we glue along $U_0^l \cap U_1^l \times \mathbb{A}^1$ to form a reduced scheme \mathcal{P} by the identification:

$$\begin{aligned}
 U_0^l \cap U_1^l \times \mathbb{A}^1 &\hookrightarrow U_k^l \times \mathbb{A}^1, \quad k = 1, 2, \\
 (x_1, \dots, x_l, \dots, x_n, \lambda) &\mapsto (x_1, \dots, x_l + (-1)^k \lambda x_l \prod_{i=1}^n x_i^{\langle u, e_i \rangle}, \dots, x_n, \lambda), \quad (4)
 \end{aligned}$$

where λ is an affine coordinate on \mathbb{A}^1 . The obvious projection onto the last component gives us a flat family $\mathcal{P} \rightarrow \mathbb{A}^1$ whose fiber over the point $\lambda = 0$ is precisely the original complete toric variety \mathbf{P}_Σ .

To describe the deformations of big and nef hypersurfaces $X \subset \mathbf{P}_\Sigma$ induced by the above deformations of the ambient toric variety, we will construct three families in $U_0^l \cap U_1^l \times \mathbb{A}^1$, $U_0^l \times \mathbb{A}^1$ and $U_1^l \times \mathbb{A}^1$, which will be patched together by the identification (4). For simplicity, we only restrict to nontrivial deformations assuming that $-u$ is the root of $U_0^l \cap U_1^l$. Let X be defined by the polynomial

$$f(x) = \sum_{m \in \Delta} a_m \prod_{i=1}^n x_i^{b_i + \langle m, e_i \rangle}$$

in S_β , where $\Delta = \Delta_D$ is the polytope in M associated to the torus invariant divisor $D = \sum_{i=1}^n b_i D_i$ and given by the conditions $b_i + \langle m, e_i \rangle \geq 0$. Consider the hypersurface in $U_0^l \cap U_1^l \times \mathbb{A}^1$ defined by the equation:

$$f_\lambda(x) := \sum_{m \in \Delta} a_m \prod_{i=1}^n x_i^{b_i + \langle m, e_i \rangle} \left(1 + c_m \lambda \prod_{i=1}^n x_i^{\langle u, e_i \rangle} \right)^{b_i + \langle m, e_i \rangle} = 0,$$

where

$$c_m = \begin{cases} 1 & \text{if } b_{l_0} + \langle m, e_{l_0} \rangle + \langle u, e_{l_0} \rangle (b_l + \langle m, e_l \rangle) > 0 \\ 0 & \text{if } b_{l_0} + \langle m, e_{l_0} \rangle + \langle u, e_{l_0} \rangle (b_l + \langle m, e_l \rangle) = 0 \\ -1 & \text{if } b_{l_0} + \langle m, e_{l_0} \rangle + \langle u, e_{l_0} \rangle (b_l + \langle m, e_l \rangle) < 0. \end{cases}$$

Then denote by $f_\lambda^k(x)$ the rational function:

$$\sum_{m \in \Delta} a_m \prod_{i=1}^n x_i^{b_i + \langle m, e_i \rangle} \left(1 + ((-1)^k + c_m) \lambda \prod_{i=1}^n x_i^{\langle u, e_i \rangle} \right)^{b_l + \langle m, e_l \rangle},$$

obtained from $f_\lambda(x)$ by the transformation (4). We claim that this function does not have poles on $U_k^l \times \mathbb{A}^1$, and, therefore, defines a hypersurface there. Indeed, the only poles can occur along the divisors D_i for $\rho_i \subset \sigma$. Since $\langle u, e_l \rangle = -1$, there are obviously no poles along D_l . Further, notice that $b_i = \langle m_\sigma, e_i \rangle$ for $\rho_i \subset \sigma$ and some $m_\sigma \in M$, by the construction of the fan Σ_X (see [M1, Section 1]). Hence, $b_{l_0} + \langle m, e_{l_0} \rangle + \langle u, e_{l_0} \rangle (b_l + \langle m, e_l \rangle) < 0$ if and only if $b_{l_i} + \langle m, e_{l_i} \rangle + \langle u, e_{l_i} \rangle (b_l + \langle m, e_l \rangle) \geq 0$ for any $0 \leq i < j$, because $b_l + \langle m, e_l \rangle + \langle u, e_l \rangle (b_l + \langle m, e_l \rangle) = 0$. If $f_\lambda^0(x)$ had a pole along D_{l_i} for $0 \leq i \leq j$, then the smallest degree in x_{l_i} would be $b_{l_i} + \langle m, e_{l_i} \rangle + \langle u, e_{l_i} \rangle (b_l + \langle m, e_l \rangle) < 0$ for some $m \in \Delta$. However, $c_m = -1$ in this case by the above, which means that $f_\lambda^0(x)$ has no poles on $U_0^l \times \mathbb{A}^1$. A similar argument shows that $f_\lambda^1(x)$ defines a hypersurface in $U_1^l \times \mathbb{A}^1$.

By the identification (4), the families given by $f_\lambda^k(x)$ in $U_k^l \times \mathbb{A}^1$, for $k = 1, 2$, fit together along $f_\lambda(x) = 0$ in $U_0^l \cap U_1^l \times \mathbb{A}^1$ to form a non-polynomial deformation $\mathcal{X} \rightarrow \mathbb{A}^1$ of the semiample hypersurface $X \subset \mathbf{P}_\Sigma$. This family is flat because algebraic families of divisors are flat (see [H, Chapter III, Example 9.8.5]).

To confirm our construction we want to compare it with the non-polynomial deformation of a Calabi-Yau hypersurface in the weighted projective space $\mathbb{P}(1, 1, 2, 2, 2)$ described in [KaKLMc].

Example 2.1. Let Y be the quasimooth Calabi-Yau hypersurface in $\mathbb{P}(1, 1, 2, 2, 2)$, with homogeneous coordinates z_1, z_2, z_3, z_4, z_5 , defined by the equation

$$z_1^8 + z_2^8 + z_3^4 + z_4^4 + z_5^4 = 0.$$

The weighted projective space $\mathbb{P}(1, 1, 2, 2, 2)$, whose fan has the following integral generators of the 1-dimensional cones

$$\{(-1, -2, -2, -2), (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}, \quad (5)$$

is singular along \mathbb{P}^2 given by $z_1 = z_2 = 0$. This singularity corresponds to the fact that the 2-dimensional cone σ generated by the first two lattice points $(-1, -2, -2, -2)$ and $(1, 0, 0, 0)$ contains another lattice point not integrally generated by these generators:

$$(0, -1, -1, -1) = \frac{1}{2}(-1, -2, -2, -2) + \frac{1}{2}(1, 0, 0, 0).$$

The crepant desingularization of $\mathbb{P}(1, 1, 2, 2, 2)$ is the toric variety \mathbf{P}_Σ whose fan is obtained from the fan determined by (5) by inserting the ray through $(0, -1, -1, -1)$. If $x_1, x_2, x_3, x_4, x_5, x_6$ denote the homogeneous coordinates of \mathbf{P}_Σ , corresponding to the five points in (5) and the sixth $(0, -1, -1, -1)$, then the crepant resolution X of Y is defined by the polynomial

$$f(x) = x_1^8 x_6^4 + x_2^8 x_6^4 + x_3^4 + x_4^4 + x_5^4.$$

To describe the non-polynomial deformation of X first embed $\mathbb{P}(1, 1, 2, 2, 2)$ into \mathbb{P}^5 by the degree 2 linear system:

$$(z_1, z_2, z_3, z_4, z_5) \mapsto (z_1^2, z_2^2, z_1 z_2, z_3, z_4, z_5).$$

Then the image of $\mathbb{P}(1, 1, 2, 2, 2)$ is the quadric $y_0 y_1 = y_2^2$, while the image of Y is the intersection of this quadric with the quartic

$$y_0^4 + y_1^4 + y_3^4 + y_4^4 + y_5^4 = 0. \quad (6)$$

The complex deformation of X , as stated in [KaKLMc], is the blow-up of the complete intersections of the quartic (6) and a family of rank 4 quadrics, like $y_0 y_1 = y_2(y_2 + 2\lambda y_3)$, in \mathbb{P}^5 along $y_0 = y_2 = 0$. We can describe this blow-up as the complete intersection

$$y_1 z_1 = (y_2 + \lambda y_3) z_2, \quad y_0 z_2 = y_2 z_1 \quad (7)$$

in $\mathbb{P}^5 \times \mathbb{P}^1$, where z_1, z_2 are the coordinates on the \mathbb{P}^1 factor. The toric variety \mathbf{P}_Σ , which is the blow-up of the quadric $y_0 y_1 = y_2^2$ along $y_0 = y_2 = 0$, is isomorphic to the complete intersection $y_1 z_1 = y_2 z_2$ and $y_0 z_2 = y_2 z_1$ under the map

$$(x_1, x_2, x_3, x_4, x_5, x_6) \mapsto (x_1^2 x_6, x_2^2 x_6, x_1 x_2 x_6, x_3, x_4, x_5) \times (x_1, x_2). \quad (8)$$

Comparing the above deformation with ours, let $l = 6$ and $l_0 = 1$. Then notice that the charts U_0^l and U_1^l , defined by $x_2 \neq 0$ and $x_1 \neq 0$ in \mathbf{P}_Σ , respectively, can be identified with the open subsets $z_2 \neq 0$ and $z_1 \neq 0$ of the complete intersection (7) by composing (8) with the maps, which send (y, z) to $(y_0 - 2\lambda y_3 \frac{z_1}{z_2}, y_1, y_2 - 2\lambda y_3, y_3, y_4, y_5, z_1, z_2)$ and $(y_0, y_1 + 2\lambda y_3 \frac{z_2}{z_1}, y_2, y_3, y_4, y_5, z_1, z_2)$, respectively. This gives the transition function between U_0^l and U_1^l :

$$(x_1, \dots, x_5, x_6) \mapsto (x_1, \dots, x_5, x_6 - 2\lambda \frac{x_3}{x_1 x_2}),$$

precisely as we had in (4) with $u = (-1, 1, 0, 0)$. Under the above identification, the quartic (6) maps to the hypersurfaces in U_0^l and U_1^l determined by the rational functions $x_1^8 x_6^4 (1 + 2\lambda \frac{x_3}{x_1 x_2}))^4 + x_2^8 x_6^4 + x_3^4 + x_4^4 + x_5^4$ and $x_1^8 x_6^4 + x_2^8 x_6^4 (1 - 2\lambda \frac{x_3}{x_1 x_2}))^4 + x_3^4 + x_4^4 + x_5^4$, which in this case coincide with our $f_\lambda^k(x)$.

3. MATCHING THE GLOBAL AND INFINITESIMAL DEFORMATIONS.

Here, we find the correspondence between the deformations constructed in the previous section and the Čech cocycles representing infinitesimal deformations of toric varieties and infinitesimal non-polynomial deformations of semiample hypersurfaces in Section 1.

To compute the infinitesimal deformation of the toric variety, we can use the formulas in [Ko, §4.2]. The open covering of each fiber in the family $\mathcal{P} \rightarrow \mathbb{A}^1$ consists of the two open sets U_0^l and U_1^l , and the transition function is given in (4). Hence, by (4.10) in [Ko], the corresponding cocycle is

$$\left\{ 2 \left(\prod_{i=1}^n x_i^{\langle u, e_i \rangle} \right) x_l \frac{\partial}{\partial x_l} \right\}_{U_0^l U_1^l}.$$

Since the identity

$$\frac{x_{l_{j-1}} \partial_{l_{j-1}}}{\text{mult}(\sigma_j)} + \frac{x_{l_{j+1}} \partial_{l_{j+1}}}{\text{mult}(\sigma_{j+1})} = \frac{\text{mult}(\sigma_j + \sigma_{j+1})}{\text{mult}(\sigma_j) \text{mult}(\sigma_{j+1})} x_{l_j} \partial_{l_j}$$

holds (see (7) in [M2]), we conclude that the above cocycle represents the same element in $H^1(\mathbf{P}_\Sigma, T_{\mathbf{P}_\Sigma})$ as the $-2\text{mult}(\sigma_j)\text{mult}(\sigma_{j+1})/\text{mult}(\sigma_j + \sigma_{j+1})$ multiple of (2) with $B = \prod_{\rho_i \subset \sigma} x_i \prod_{i=1}^n x_i^{\langle u, e_i \rangle}$ does.

For the hypersurface $X \subset \mathbf{P}_\Sigma$, the infinitesimal deformation corresponding to the flat family $\mathcal{X} \rightarrow \mathbb{A}^1$ can be found using [Ar] and [H, Chapter III, Section 9]. We replace the base \mathbb{A}^1 by $\text{Spec}(\mathbb{C}[\varepsilon])$ (abusing notation, $\mathbb{C}[\varepsilon]$ denotes $\mathbb{C}[\varepsilon]/\langle \varepsilon^2 \rangle$): take the morphism

$$\text{Spec}(\mathbb{C}[\varepsilon]) \rightarrow \text{Spec}\mathbb{C}[\lambda] \cong \mathbb{A}^1,$$

which arises from the ring homomorphism that sends λ to ε , then by base extension we obtain another family

$$\mathcal{X}' = \mathcal{X} \times_{\mathbb{A}^1} \text{Spec}(\mathbb{C}[\varepsilon])$$

flat over the new base. It is not difficult to see that this family is glued from the hypersurfaces

$$f_\varepsilon^k(x) = \sum_{m \in \Delta} a_m \prod_{i=1}^n x_i^{b_i + \langle m, e_i \rangle} \left(1 + \varepsilon((-1)^k + c_m)(b_l + \langle m, e_l \rangle) \prod_{i=1}^n x_i^{\langle u, e_i \rangle} \right) = 0$$

in $U_k^l \times \text{Spec}(\mathbb{C}[\varepsilon])$ along the hypersurface

$$f_\varepsilon(x) = \sum_{m \in \Delta} a_m \prod_{i=1}^n x_i^{b_i + \langle m, e_i \rangle} \left(1 + \varepsilon c_m(b_l + \langle m, e_l \rangle) \prod_{i=1}^n x_i^{\langle u, e_i \rangle} \right) = 0$$

in $U_0^l \cap U_1^l \times \text{Spec}(\mathbb{C}[\varepsilon])$ with the identification:

$$U_0^l \cap U_1^l \times \text{Spec}(\mathbb{C}[\varepsilon]) \hookrightarrow U_k^l \times \text{Spec}(\mathbb{C}[\varepsilon]), \quad k = 1, 2,$$

which is defined by sending a function $a + b\varepsilon$ on the affine scheme $U_k^l \cap \mathbf{A}_\tau \times \text{Spec}(\mathbb{C}[\varepsilon])$, where $\mathbf{A}_\tau = \text{Spec}(S_\tau)_0$ is the affine toric variety associated with a cone $\tau \in \Sigma$ and S_τ is the localization of S at $\prod_{\rho_i \not\subset \tau} x_i$ (see [C1, Lemma 2.2]), to the function

$$a + \left(b + (-1)^k \left(\prod_{i=1}^n x_i^{\langle u, e_i \rangle} \right) x_l \partial_l a \right) \varepsilon \tag{9}$$

on the affine scheme $U_0^l \cap U_1^l \times \text{Spec}(\mathbb{C}[\varepsilon])$.

To find the Čech cocycle corresponding the infinitesimal deformation we first build trivializations:

$$\phi_{(i_0, k_0)} : X \cap U_{i_0} \cap U_{k_0}^l \cap \mathbf{A}_\tau \times \text{Spec}(\mathbb{C}[\varepsilon]) \cong \mathcal{X}' \cap U_{i_0} \cap U_{k_0}^l \cap \mathbf{A}_\tau \times \text{Spec}(\mathbb{C}[\varepsilon])$$

by mapping a function $a + b\varepsilon$ on the target scheme to the function

$$a + \left(b - \left(\prod_{i=1}^n x_i^{\langle u, e_i \rangle} \right) \sum_{m \in \Delta} a_m ((-1)^{k_0} + c_m) \left(x_l \partial_l \prod_{i=1}^n x_i^{b_i + \langle m, e_i \rangle} \right) \frac{\partial_{i_0}}{f_{i_0}} a \right) \varepsilon \tag{10}$$

on the first scheme. It is straightforward to check that this map is a well-defined isomorphism of the rings of functions, which sends f_ε^k/x^D to f/x^D where $x^D \in S_\beta$ is a monomial invertible in S_τ (see [BC, page 317]). The inverse of $\phi_{(i_0, k_0)}$ is given by changing the sign after b in (10). Then, taking into account (9), the automorphisms $\phi_{(i_0, k_0)}^{-1} \circ \phi_{(i_1, k_1)}$ on

$$X \cap U_{i_0} \cap U_{k_0}^l \cap U_{i_1} \cap U_{k_1}^l \cap \mathbf{A}_\tau \times \text{Spec}(\mathbb{C}[\varepsilon])$$

correspond on the level of rings to mapping a function $a+b\varepsilon$ to $a+(b+\theta_{(i_0,k_0)(i_1,k_1)}a)\varepsilon$, where $\theta_{(i_0,k_0)(i_1,k_1)}$ is the polyvector field

$$\begin{aligned} & \left(\prod_{i=1}^n x_i^{\langle u, e_i \rangle} \right) \left(((-1)^{k_0} - (-1)^{k_1}) x_l \partial_l \right. \\ & \quad \left. - \sum_{m \in \Delta} a_m \left(x_l \partial_l \prod_{i=1}^n x_i^{b_i + \langle m, e_i \rangle} \right) \left(((-1)^{k_0} + c_m) \frac{\partial_{i_0}}{f_{i_0}} - ((-1)^{k_1} + c_m) \frac{\partial_{i_1}}{f_{i_1}} \right) \right) \end{aligned}$$

The Čech cocycle $\{\theta_{(i_0,k_0)(i_1,k_1)}\}_{(i_0,k_0)(i_1,k_1)}$ represents the infinitesimal deformation \mathcal{X}' of the hypersurface X in $H^1(X, \mathcal{T}_X)$. It is not hard to see that, up to a polynomial infinitesimal deformation γ_A for appropriate $A \in S_\beta$, the cocycle

$$-2 \frac{\text{mult}(\sigma_j) \text{mult}(\sigma_{j+1})}{\text{mult}(\sigma_j + \sigma_{j+1})} (\gamma_B^l)_{(i_0,j_0)(i_1,j_1)}$$

with $B = \prod_{\rho_i \subset \sigma} x_i \prod_{i=1}^n x_i^{\langle u, e_i \rangle}$ corresponds to the same element in $H^1(X, \mathcal{T}_X)$ as $\{\theta_{(i_0,k_0)(i_1,k_1)}\}_{(i_0,k_0)(i_1,k_1)}$ does.

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